University of Applied Sciences Northwestern Switzerland
School of Business

Vague Knowledge: Fuzzy Logic

Acknowledgement

- Using slides also from Holger Wache and Gwendolin Wilke

Repetition: Reliability of Knowledge

- Exact knowledge:
  - „It is raining."
- Uncertain knowledge:
  - „I believe it will not rain tomorrow.“
- Incomplete knowledge:
  - „The temperature ist between 10 and 15 degree Celsius“
  - "It will rain between 2 and 5 mm tomorrow"
- Vague knowledge (interpretation-dependent knowledge):
  - „The weather is good."

Application

- Definition of rules with fuzzy values
- Example:
  - Fuzzy-Logic Controller for a heating controller

    IF Temperature = normal AND humidity = high
    THEN heating power = high

  - Product recommendation

    IF requirement = normal AND price = low
    THEN product = standard

FUZZY SETS

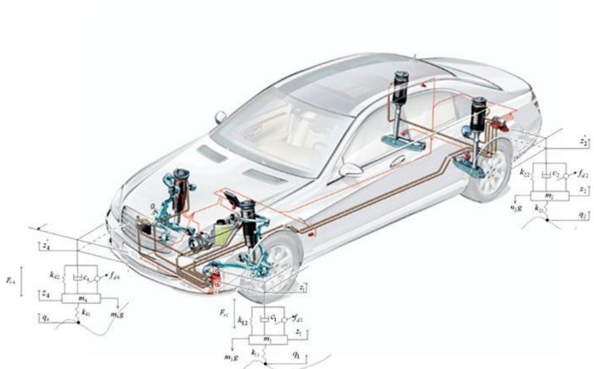
Applications of Fuzzy Logic

- Fuzzy Systems became well-known as control systems (Washing machine, …)
- Defining rules with vague knowledge

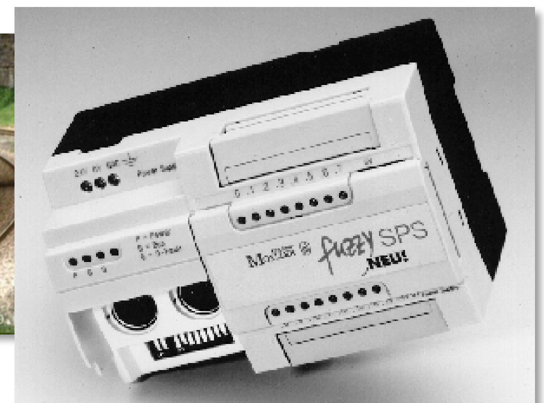
  IF Temperature = normal AND Humidity = high
  THEN HeatingPower = high

- Other application areas: Diagnosis, Language understanding, …

Washing Machine



Active Suspension Control System



Inventor of Fuzzy Logic

Lotfi Zadeh 2010

Lotfi Zadeh 1945

Classical vs. Fuzzy Sets

- Bald Men Paradox:
  - Would you describe a man with 1 hair on his head as bald? YES.
  - Would you describe a man with 2 hairs on his head as bald? YES.
  - Would you describe a man with 3 hairs on his head as bald? YES.

  ....

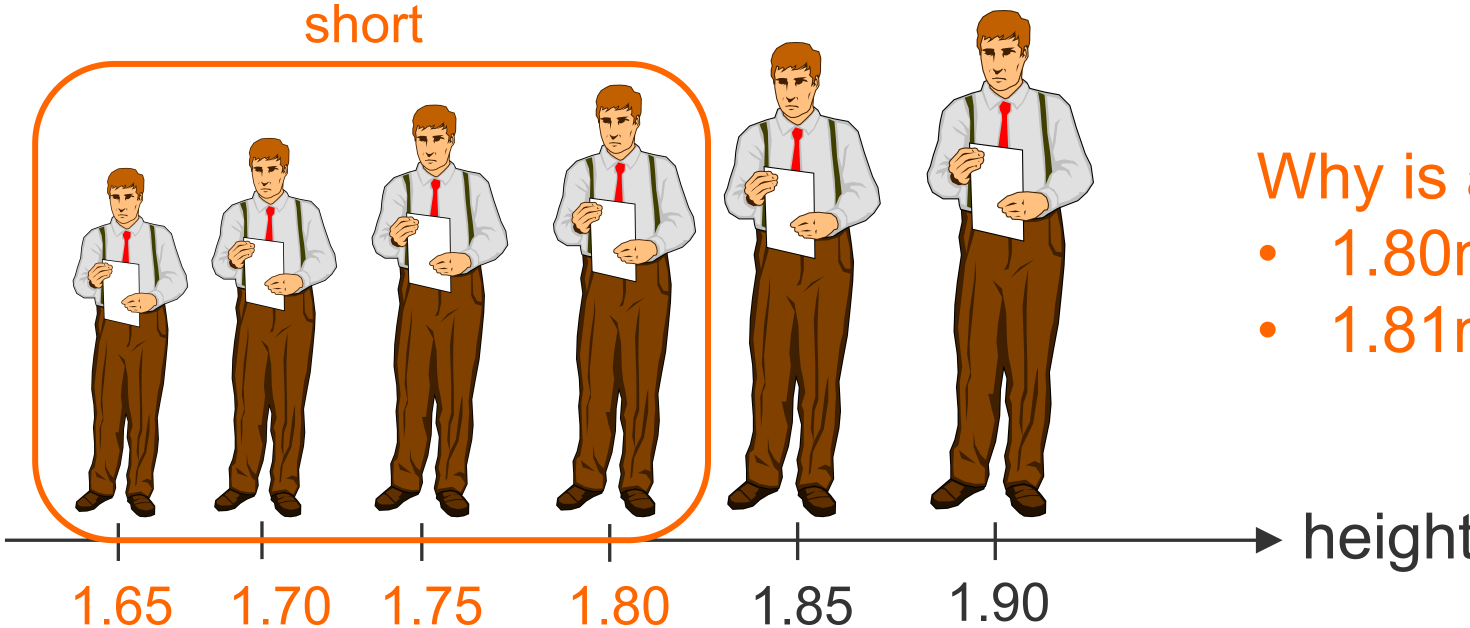
  - Would you describe a man with 10000 hairs on his head as bald? NO.

Where to draw the line?

Who is short and who is tall? And who is medium?

Classical vs. Fuzzy Sets

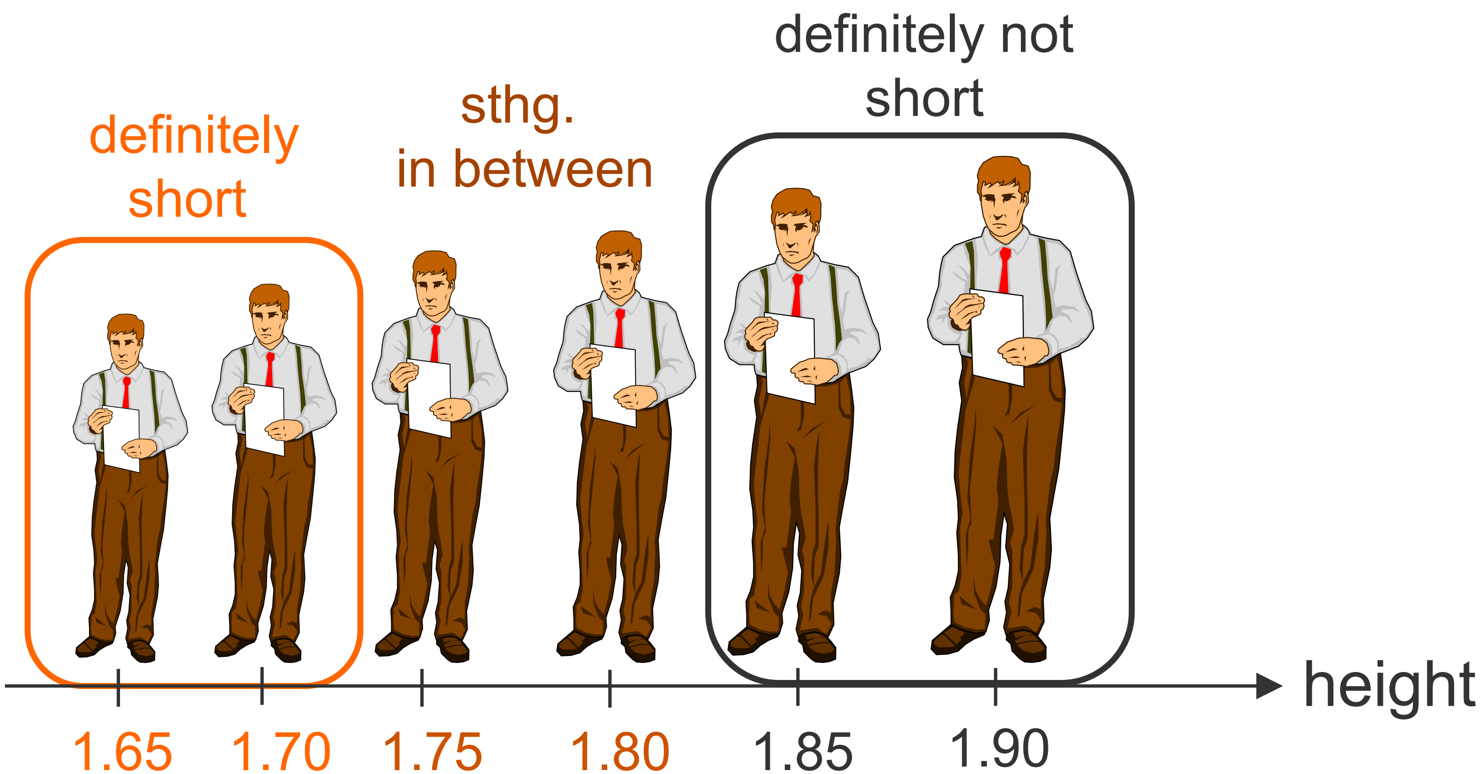
- When is a man short?
- Classical Set Theory: Either short or not short.
  E.g.: set of short men S= {m | height(m) ≤ 1.80}



Why is a man of
- 1.80m short
- 1.81m not short?

Classical vs. Fuzzy Sets

- Fuzzy sets have unsharp boundaries:

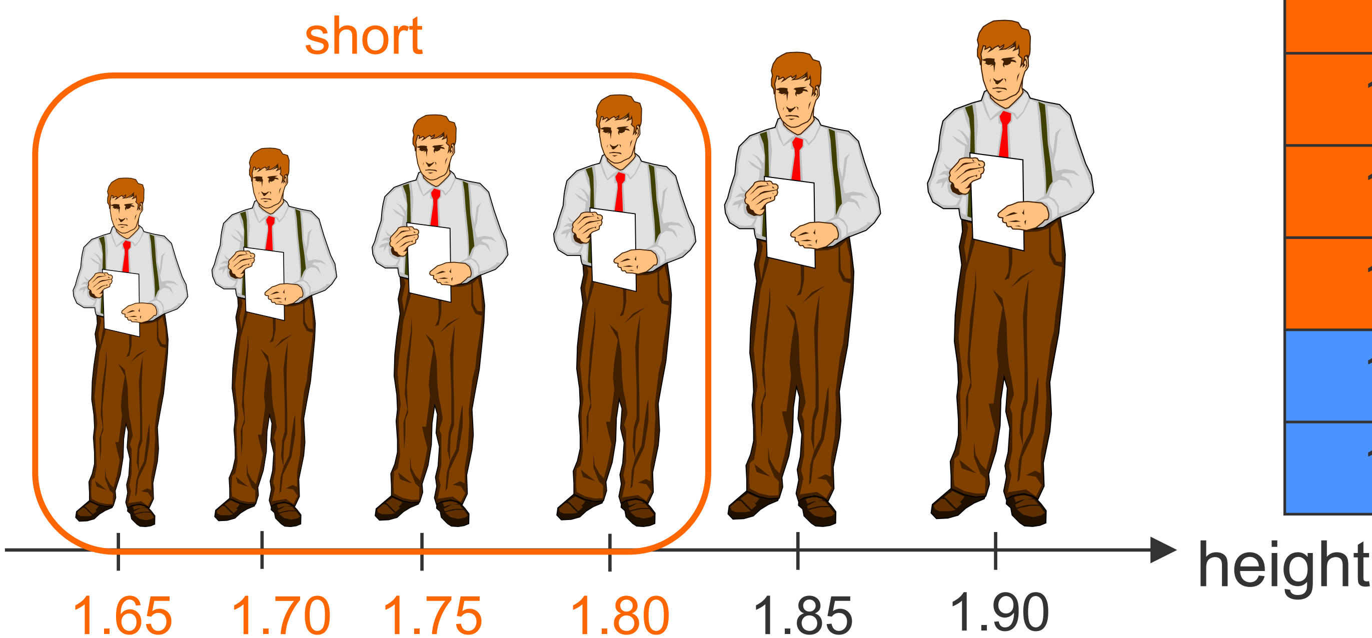


Classical vs. Fuzzy Sets

- A classical set can be seen as a special case of a fuzzy set, where the fuzziness of the set boudary is infinitely small.
- Classical sets are also called crisp sets.

Representation by membership function

Classical sets, e.g.: set of short men S= {m|height(m) ≤ 1.80}

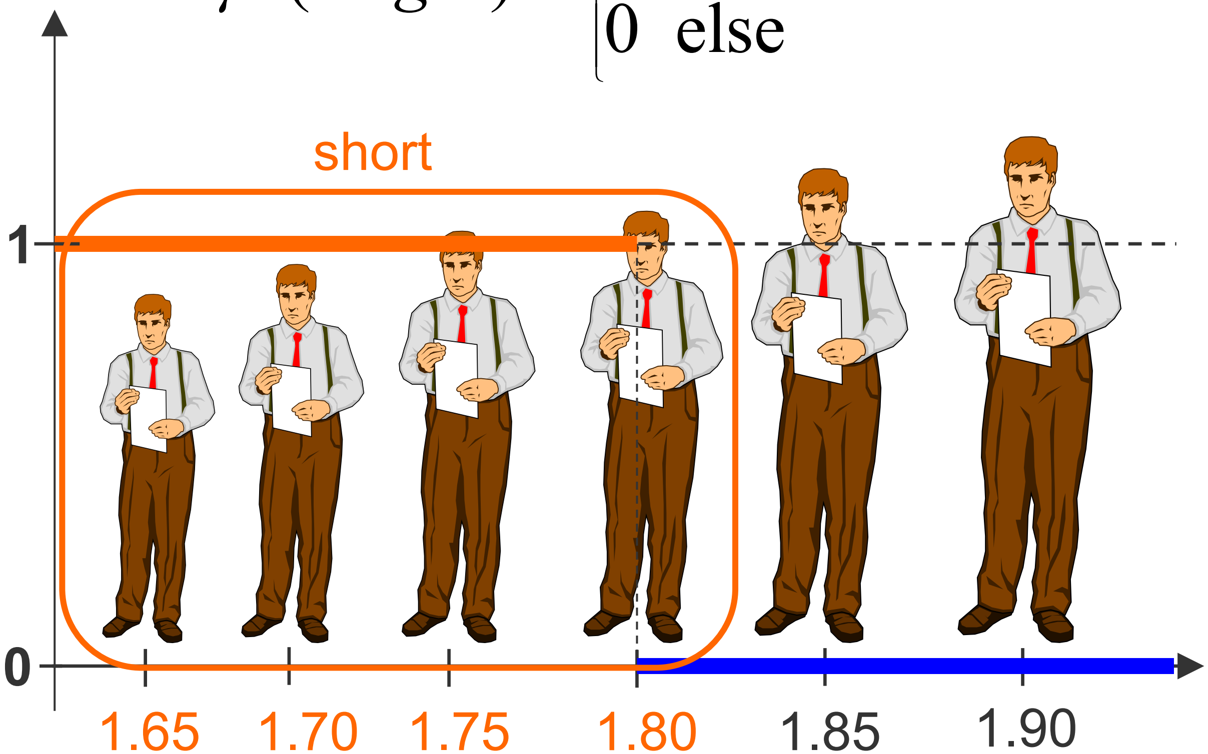


height	*short?*
1.65	1
1.70	1
1.75	1
1.80	1
1.85	0
1.90	0

Representation by membership function

Classical sets, e.g.: set of short men S= {m | height(m) ≤ 1.80}

$$\mu_S(height) = \begin{cases} 1 & \text{if } hight \leq 1.80 \\ 0 & \text{else} \end{cases}$$



height	*short?*
1.65	1
1.70	1
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1.90	0

Representation by membership function

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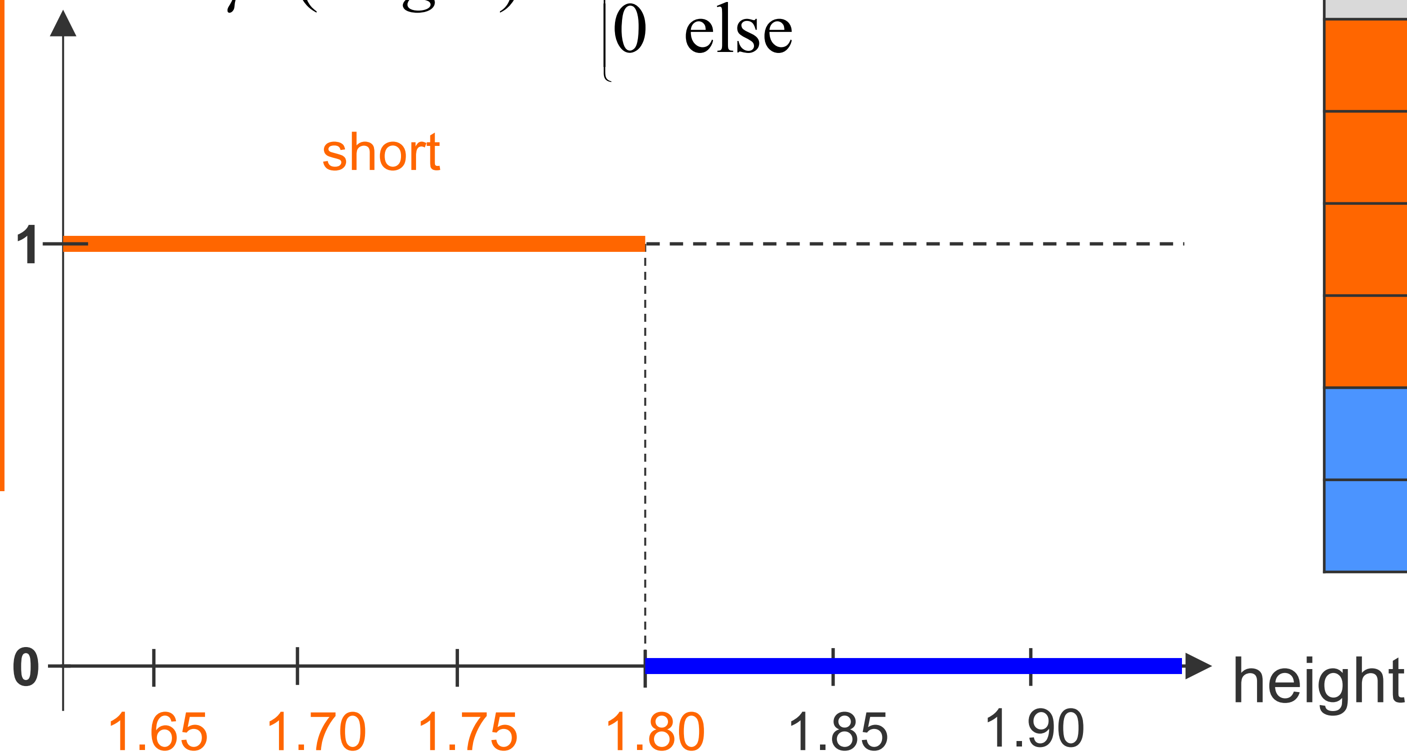
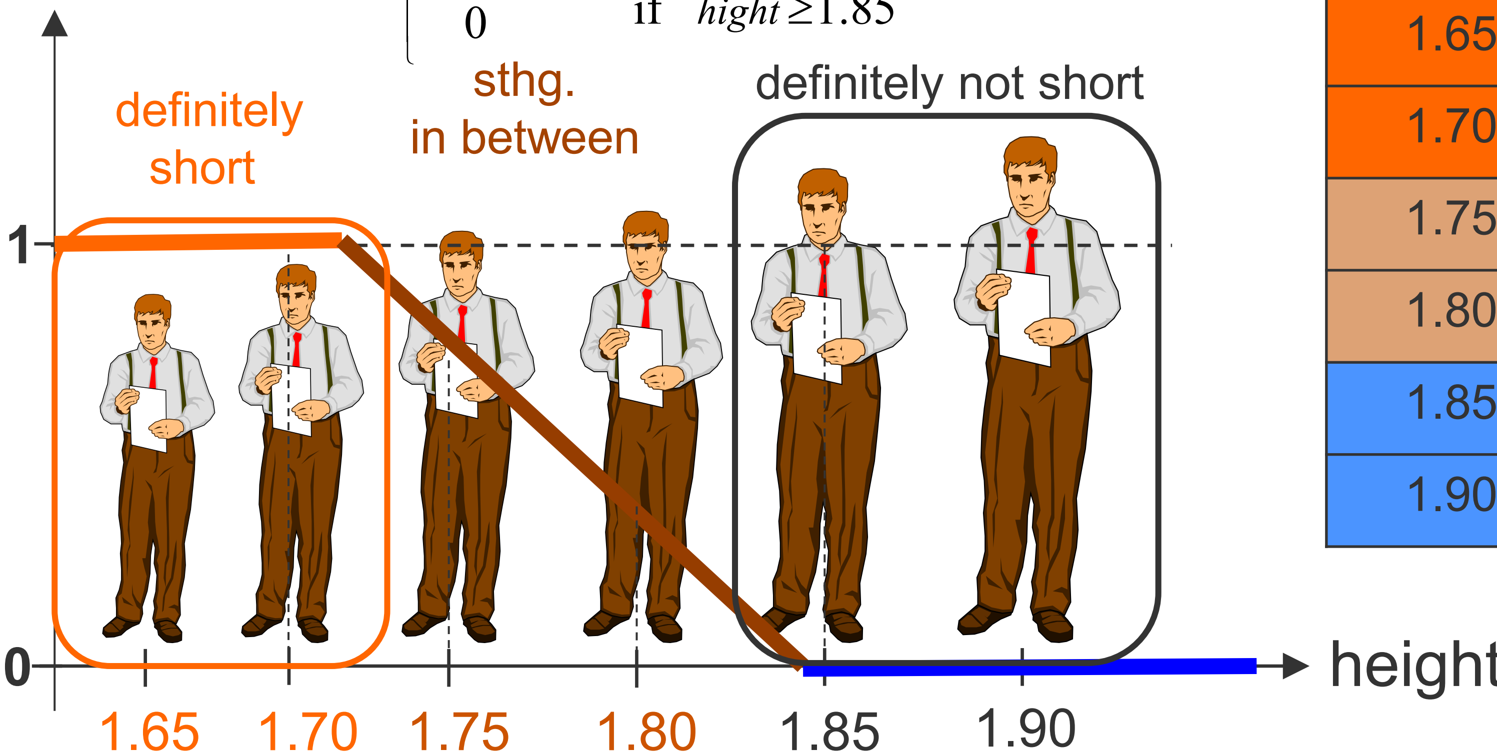


height	*short?*
1.65	1
1.70	1
1.75	1
1.80	1
1.85	0
1.90	0

Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(hight) = \begin{cases} 1 & \text{if} \quad hight \le 1.70 \\ \dfrac{1.85 - hight}{1.85 - 1.70} & \text{if} \quad 1.70 \le hight \le 1.85 \\ 0 & \text{if} \quad hight \ge 1.85 \end{cases}$$

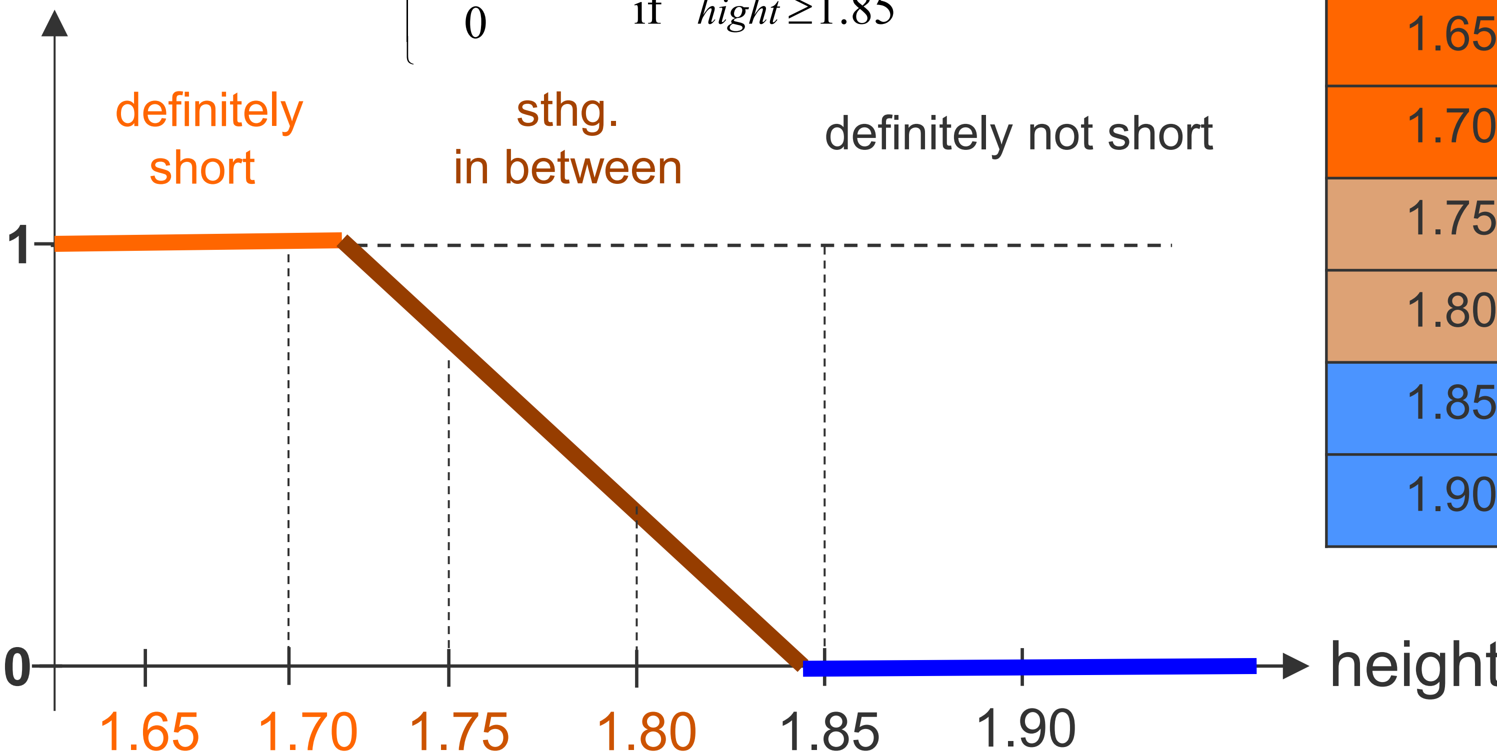


height	*short?*
1.65	1
1.70	1
1.75	2/3
1.80	1/3
1.85	0
1.90	0

Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(hight) = \begin{cases} 1 & \text{if } hight \leq 1.70 \\ \frac{1.85 - hight}{1.85 - 1.70} & \text{if } 1.70 \leq hight \leq 1.85 \\ 0 & \text{if } hight \geq 1.85 \end{cases}$$

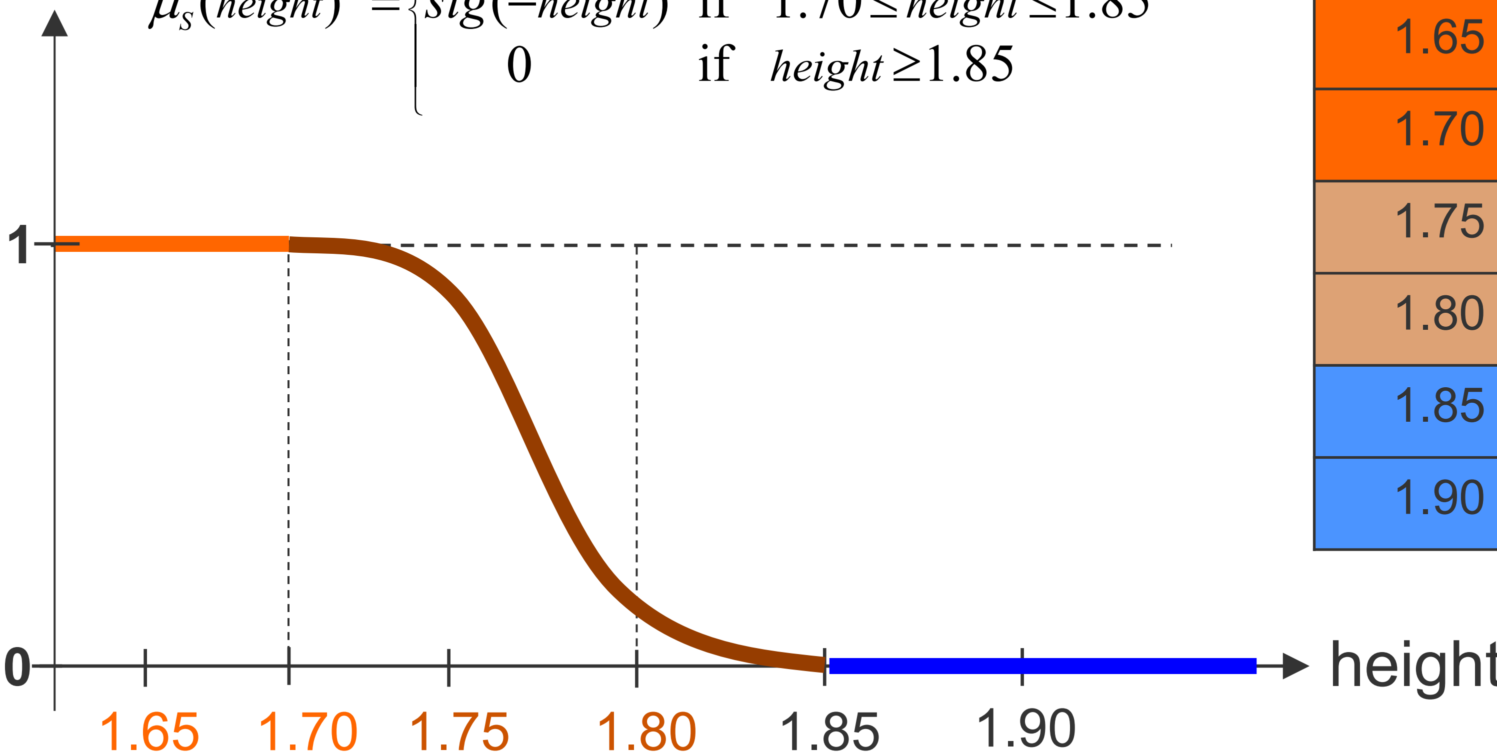


height	*short?*
1.65	1
1.70	1
1.75	2/3
1.80	1/3
1.85	0
1.90	0

Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(height) = \begin{cases} 1 & \text{if } height \le 1.70 \\ sig(-height) & \text{if } 1.70 \le height \le 1.85 \\ 0 & \text{if } height \ge 1.85 \end{cases}$$



height	*short?*
1.65	1
1.70	1
1.75	
1.80	
1.85	0
1.90	0

Exercise: Fuzzy Sets for Size of People

- Draw fuzzy sets for short, medium and tall men; use trapezoidal membership functions.
- Here are the restrictions:
  - Men below 1.60 are definitely short
  - Men taller than 175 are definitely not short
  - Men taller than 190 are definitely tall
  - Men smaller than 180 are not tall
  - Men between 170 and 185 are medium
  - Men below 165 are not medium
  - Men taller than 190 are not medium

FUZZY SET THEORY

Fuzzy Set Theory

Operations on Fuzzy Sets:

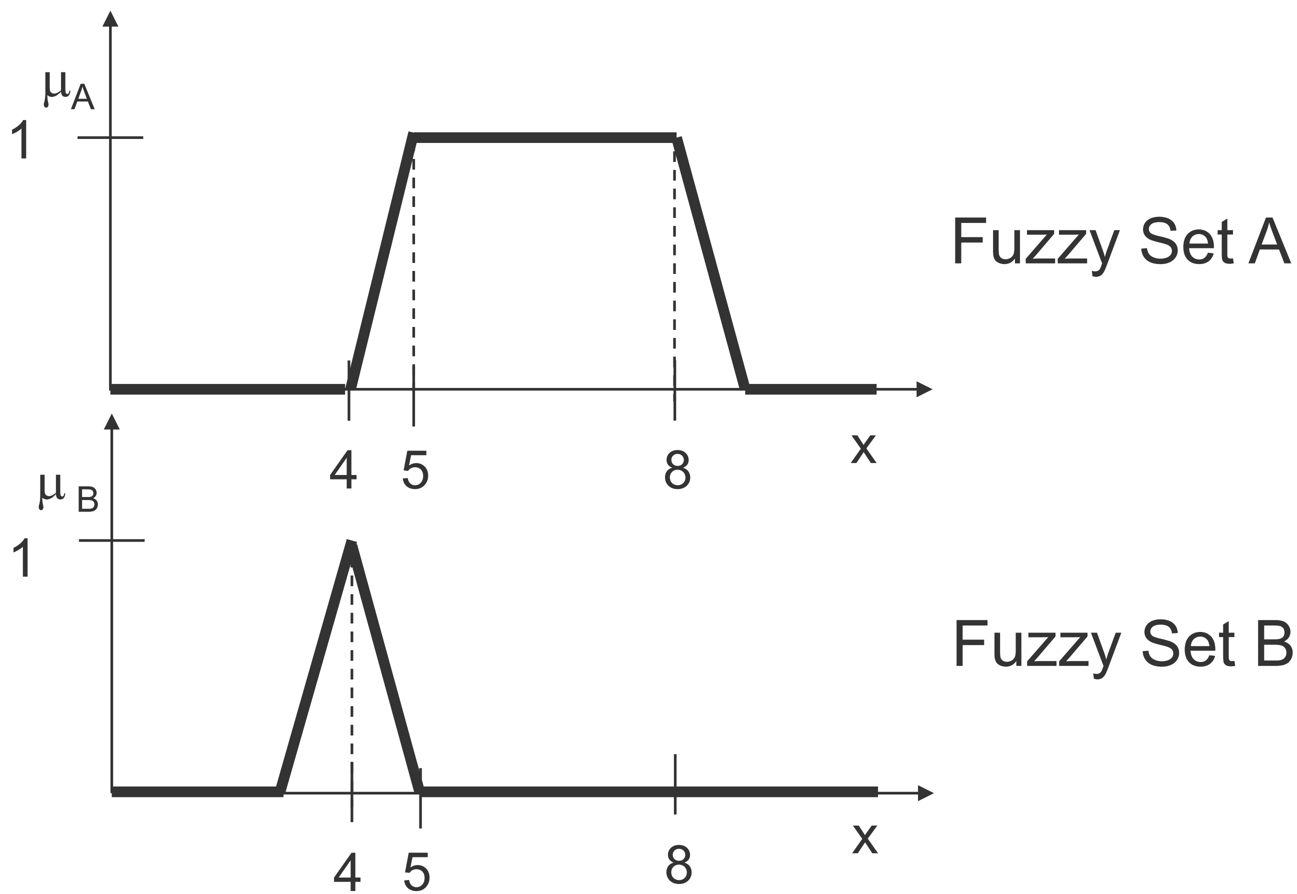
For Fuzzy Sets we can define operations

- ***intersection***,
- ***union***
- ***negation***

… analogue to classical sets.

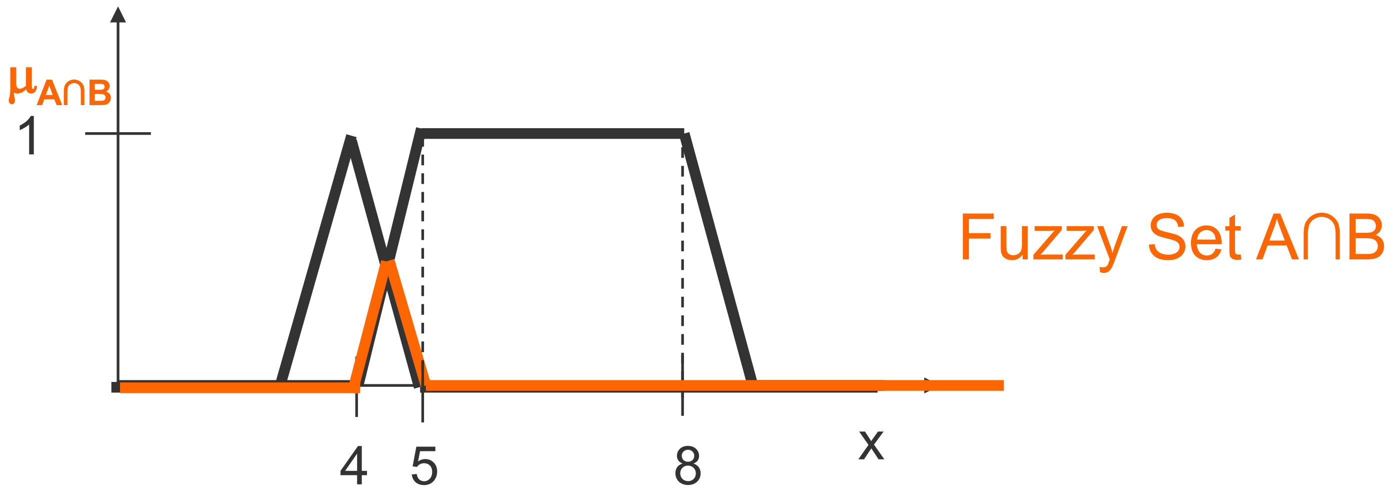
Operations with Fuzzy Sets

Intersection:



Operations with Fuzzy Sets

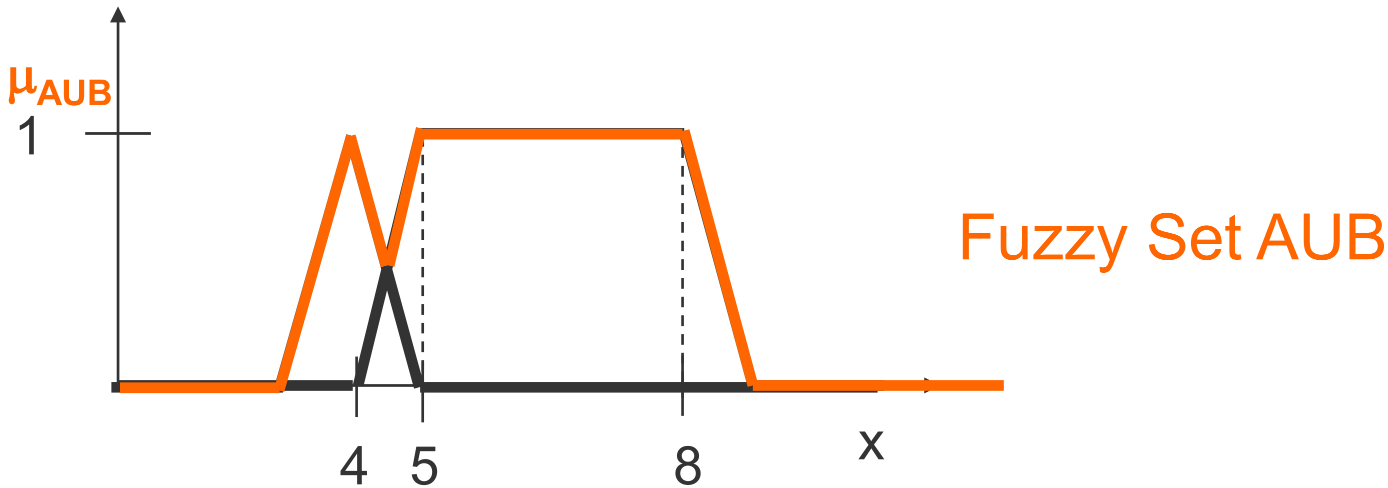
Intersection: AND



Minimum Operator: $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

Operations with Fuzzy Sets

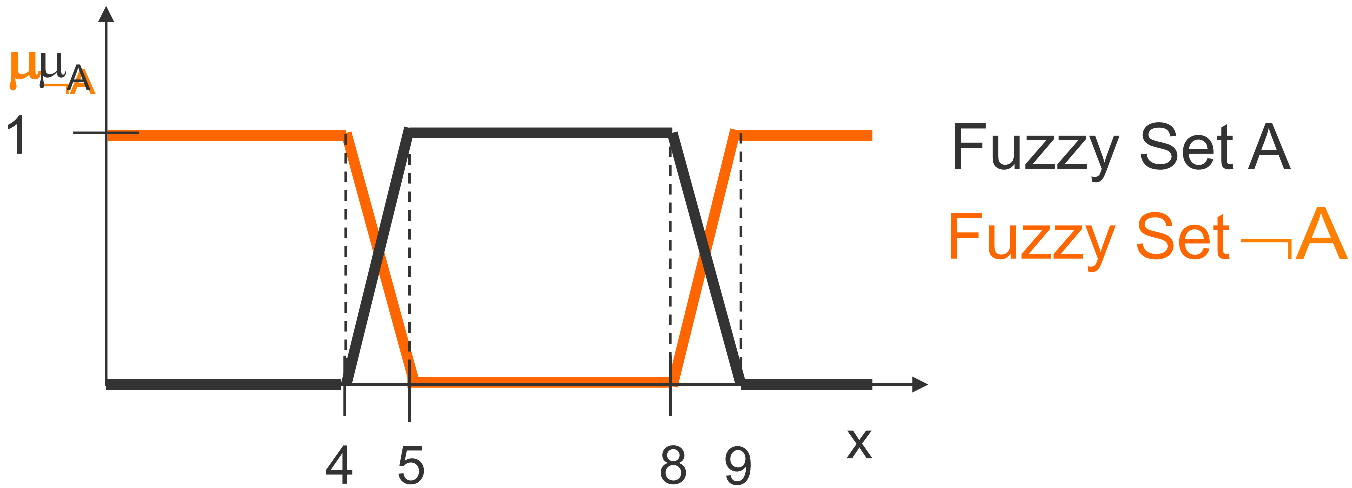
Union:



Maximum Operator: $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$

Operations with Fuzzy Sets

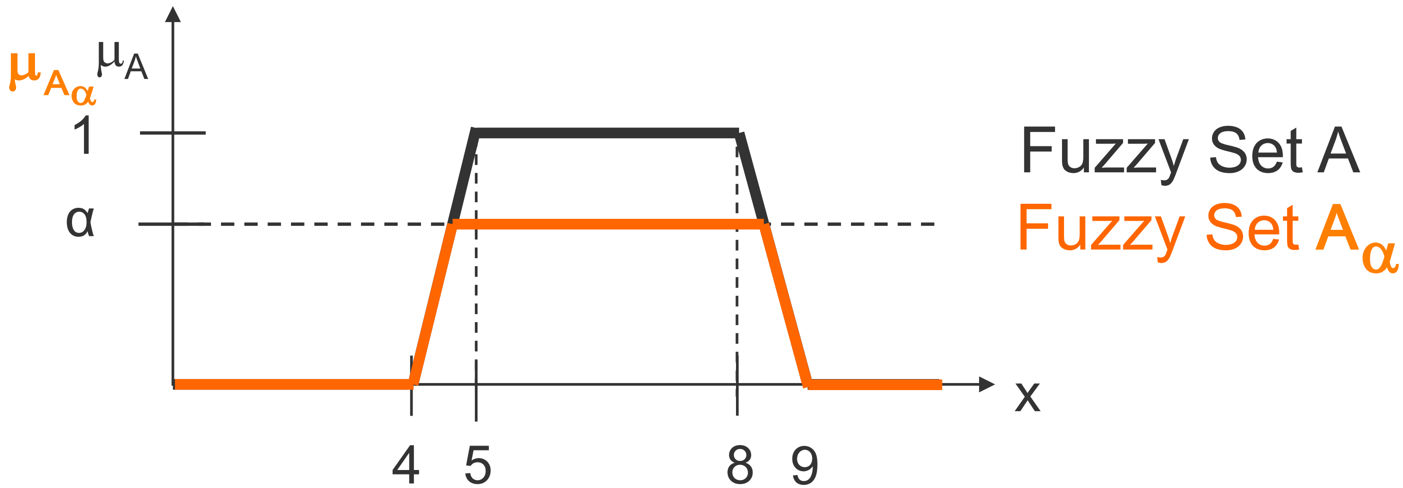
Negation:



Complement Operator: $\mu_{\neg A}(x) = 1 - \mu_A(x)$

Operations with Fuzzy Sets

Alpha-cut:



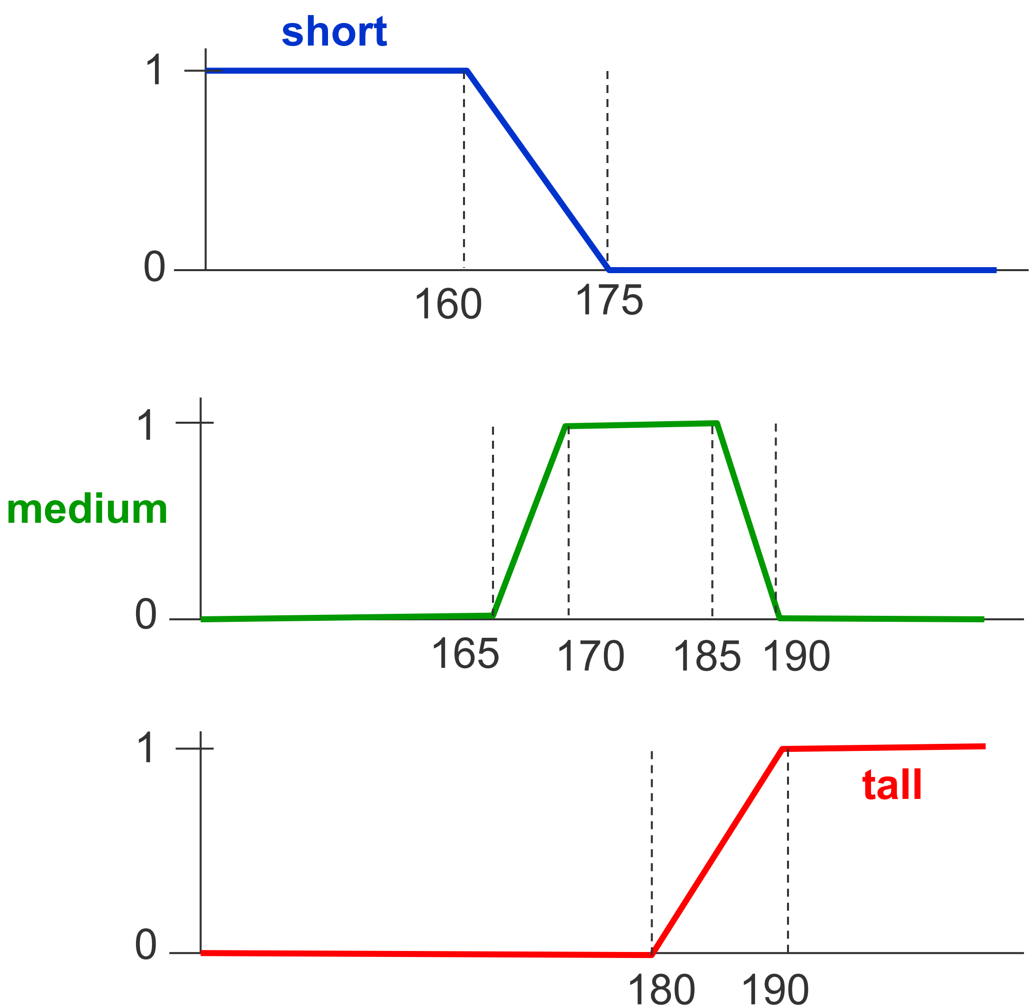
α-Cut Operator: $\mu_{A_\alpha}(x) = \min\{\mu_A(x), \alpha\}$

Exercise: Fuzzy Sets for Size of People (2)

- Draw the following fuzzy sets of people:
  - NOT short
  - NOT medium
  - NOT tall
  - Short UNION(OR) NOT tall
  - NOT short INTERSECTION(AND) NOT tall

- Is (NOT Short INTERSETION(AND) NOT tall) = medium?

Solution: Fuzzy Sets for Size of People (2)



not short

not medium

not tall

FUZZY LOGIC

Fuzzy Logical Operators

- They modify or combine fuzzy logical statements.
  - E.g.: AND, OR, NOT, …
- They are operations on membership degrees:
  - AND: minimum, $\mu_{A \wedge B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$
  - OR: maximum, $\mu_{A \vee B}(x,y) = \max\{\mu_A(x), \mu_B(y)\}$
  - NOT: complement $\mu_{\neg A}(x) = 1 - \mu_A(x)$
- Note: There are serveral possibilities to define fuzzy logic operators! We use the above.

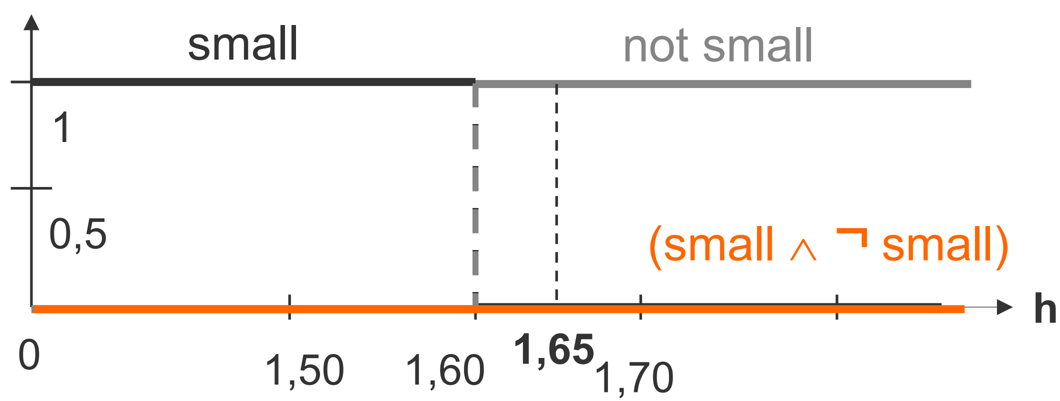
Fuzzy Logic „Paradox“

In classical logic, a statement and its negation cannot be true at the same time:

$$(s \ni \neg s) = 0$$

„Tertium non datur“
(law of the excluded middle)

Example: Classical statement s=„Bob is small“,
where *small* is specified by the following crisp set:



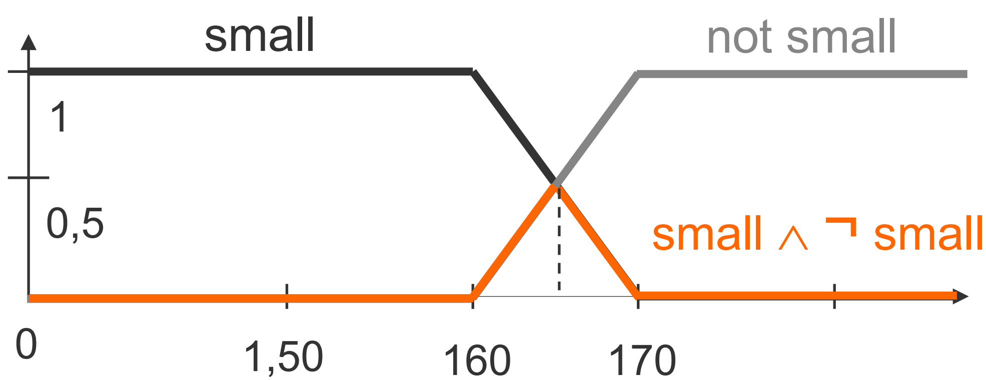
If height(Bob)=1.65, then $(s \wedge \neg s) = \min\{0,1\}=0$.

Fuzzy Logic „Paradox“

In fuzzy logic, a statement and ist negation can both be (partially) true at the same time:

$(s \wedge \neg s) \neq 0$ for some s

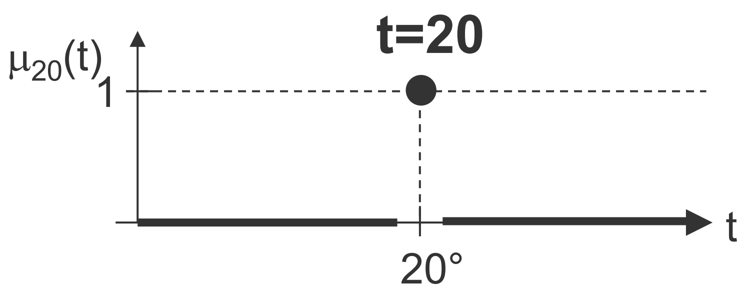
Example: Fuzzy statement s= „Bob is small“,
where small is specified by the following fuzzy set:



If height(Bob)=1.65, then $(s \wedge \neg s) = \min\{0.5,0.5\}=0.5$

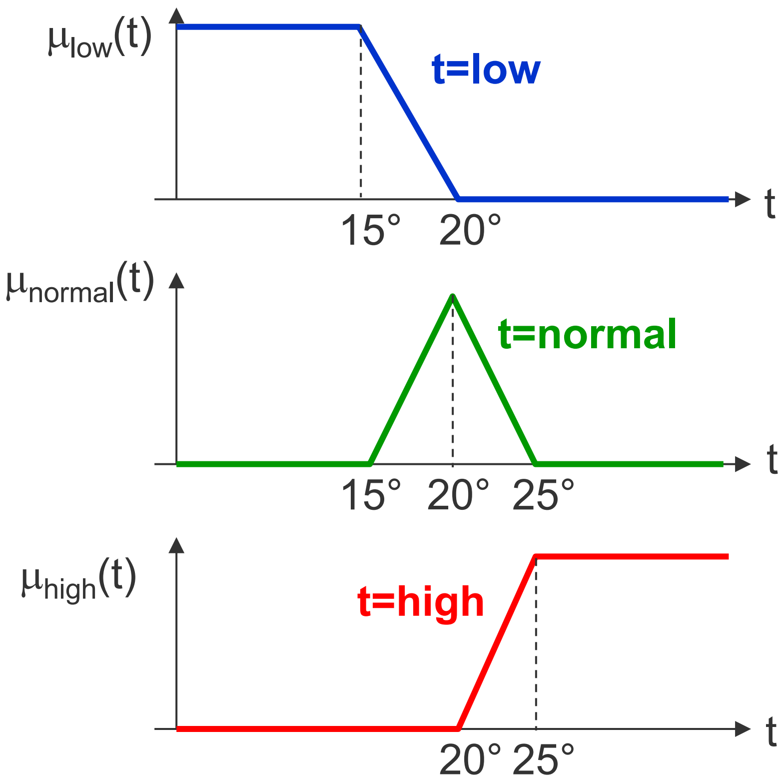
Classical vs. Linguistic Variables

Example: Classical variable «**temperature**» (t).
t takes exact values in the interval **[-50,50]**,e.g., t=20:

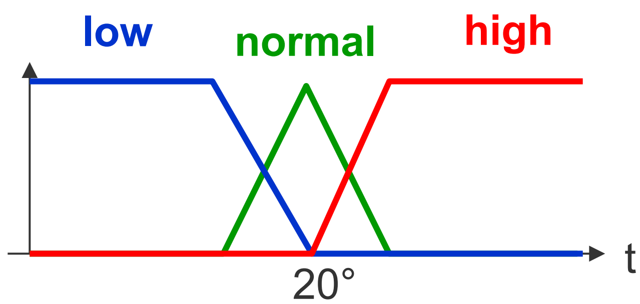


Classical vs. Linguistic Variables

Example: Linguistic variable «**temperature**» (t).
t takes the fuzzy values ***low, normal, high***, *e.g., t=low.*
Fuzzy values are defined as **Fuzzy Sets**:



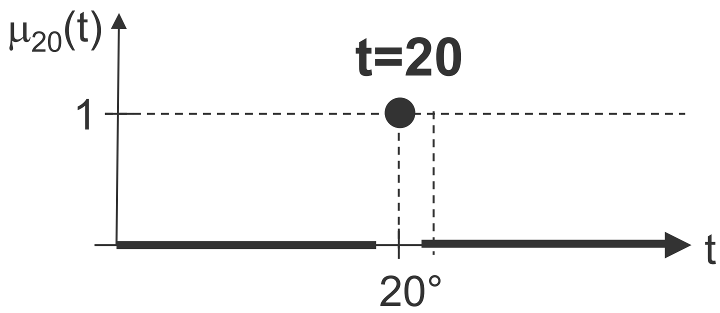
In one graphic:



Classical Logical Statements

The possible truth values of an exact statement are:
1 (True) or 0 (False).

Example: Exact statement s=«The temperature is 20°C.»



«Temperature» is a classical variable (t). Takes the value t=20.

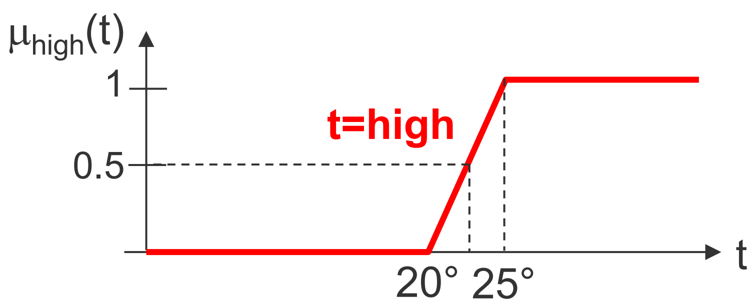
Assume the temperature is 22.5°C.

Then the truth value of s is 0.

Fuzzy Logical Statements

The possible truth values of a fuzzy statement are 1 (True), 0 (False), and every value in between.

Example: Fuzzy statement s=«The temperature is high.»



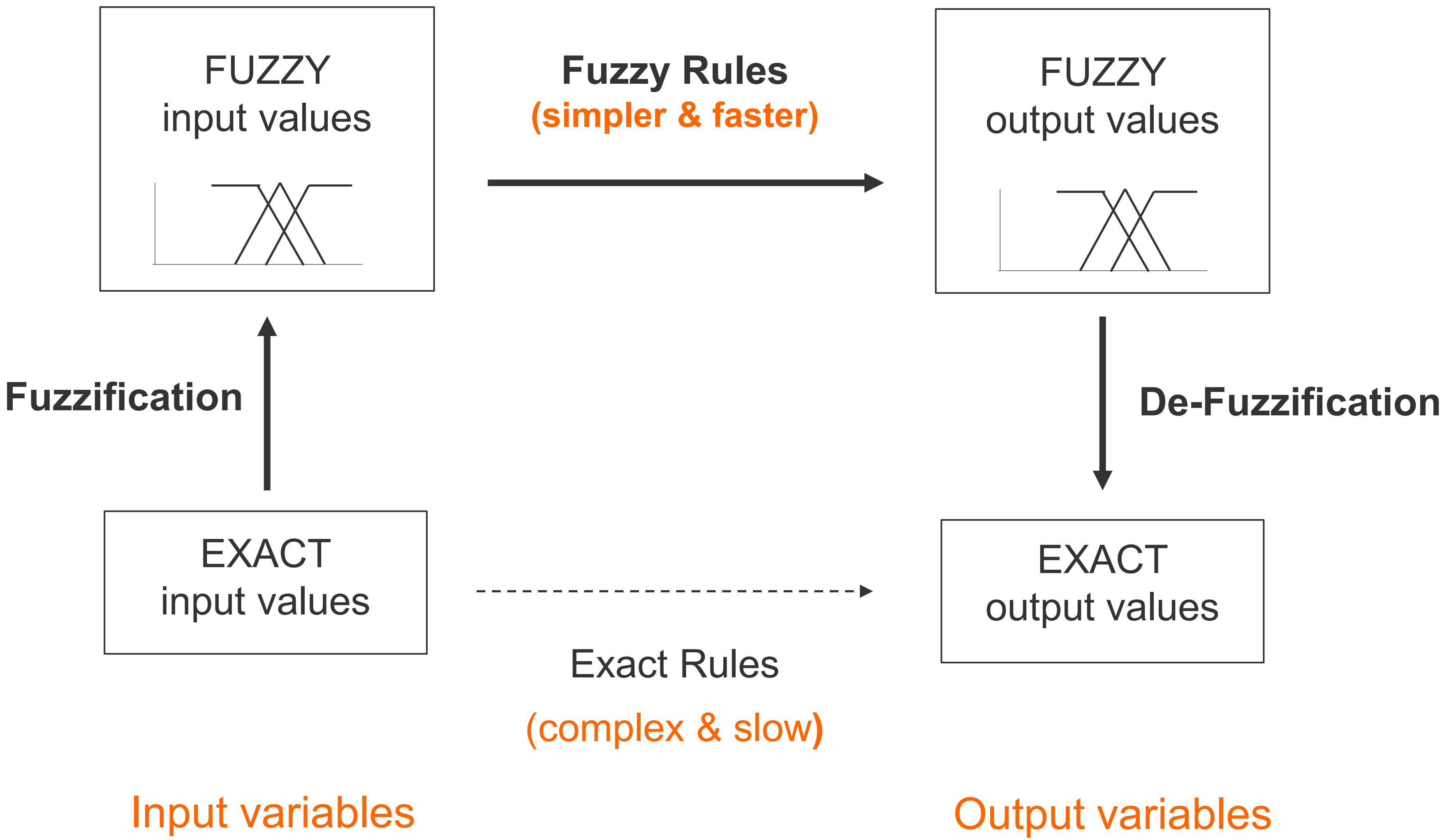
«Temperature» is a linguistic valriable (t). Takes the value t=high.

Assume the temperature is 22.5°C. Then the truth value of s is 0.5.

The truth value of a fuzzy statement is also called truth degree. The truth degree indicates the degree of compatibility of the exact value 22.5°C with the fuzzy statements s.

APPLICATIONS OF FUZZY LOGIC

Designing a Fuzzy Controller (Procedure)



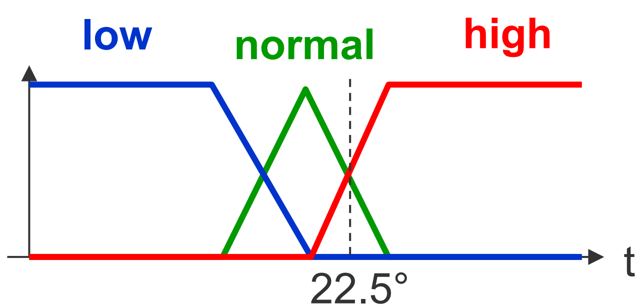
Fuzzification

- Transformation of exact variables to linguistic variables, and
- Transformation of exact values to fuzzy values (fuzzy sets).

Example: Fuzzification of variable «temperature»:

$$t \in [-50,50] \;\rightarrow\; t \in \{low,\ normal,\ high\}$$

$$t = 22.5°C \;\rightarrow\; \{\mu_{low}(t)=0,\ \mu_{normal}(t)=0.5,\ \mu_{high}(t)=0.5\}$$



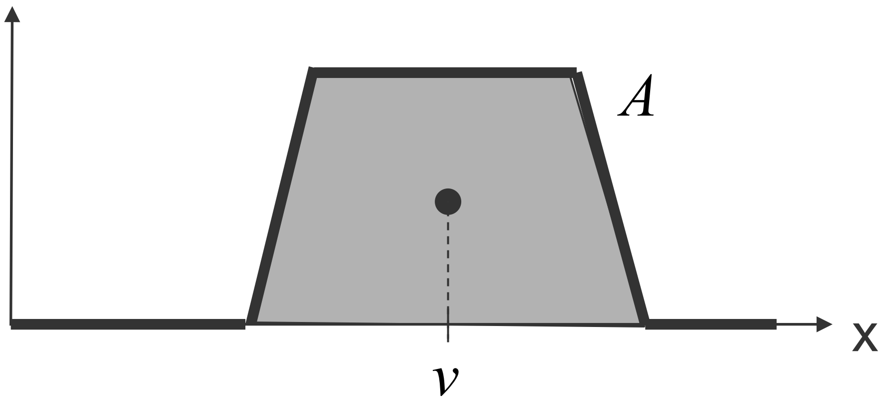
Defuzzification

= Transformation of a fuzzy set to an exact value (number).

Different possible methods, e.g.,

- Center of gravity method
- Maximum method
- Weighted average method

Example: Centre of gravity method (Sugeno 1985, most commonly used):



$$v = \frac{\int \mu_A(x) \cdot x \, dx}{\int \mu_A(x) \, dx}$$

Disadvantage: Computationally difficult for complex membership functions.

Example: Fuzzy Logic Controller

- **Problem:** Car heating system
  - The heating systems of a car should keep the temperature constant.
  - The heating power that is necessary depends on the **temperature** and the **air humidity** in the car:
    - The *higher* the temperature, the *lower* must be the heating power.
    - The *lower* the temperature, the *higher* must be the heating power.
    - The humidity interacts with temperature.
  - Sensors show the current temperature and humidity.

Fuzzy Logic Controller „Car heating system”

Steps to build the fuzzy controller

1. Specify Input and Output variables
2. Fuzzification of variables and values
3. Define fuzzy rules
4. Defuzzification

Fuzzy Logic Controller „Car heating system"

Step 1: Specify Input and Output variables

humidity → heating power
temperature → heating power

Input variables: humidity, temperature

Output variables: heating power

Fuzzy Logic Controller „Car heating system”

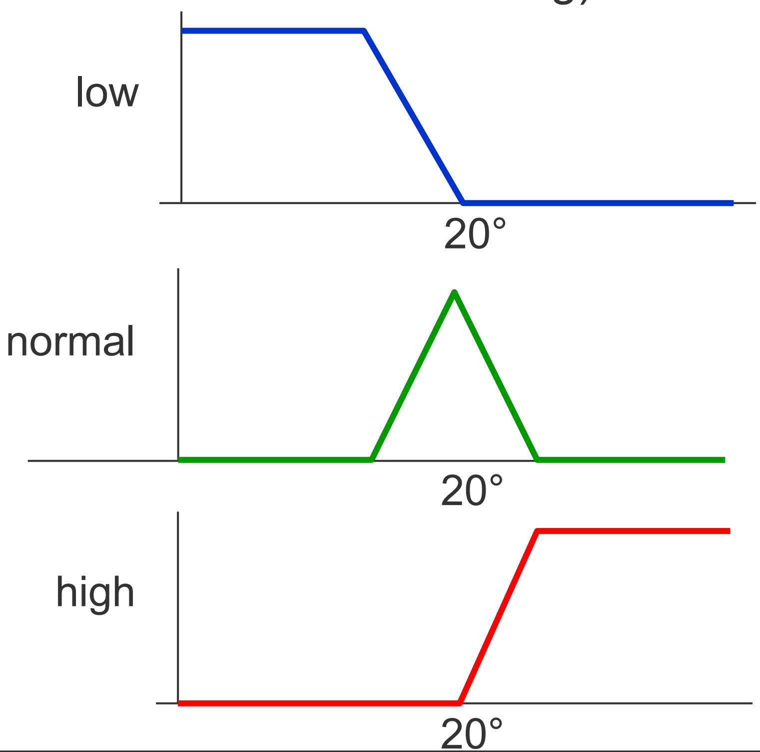
Step 2a: Fuzzification of variables and values:

- Determine linguistic variables:
  - Humidity: {low, medium, high}
  - Temperature: {low, normal, high}
  - Heating power: {low, normal, increased, high}
- Specify the fuzzy values of the linguistic variables as fuzzy sets
  - see next slide!

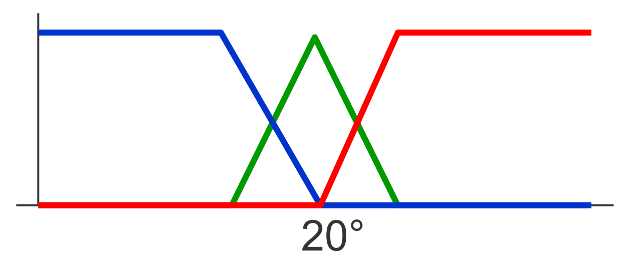
Fuzzy Logic Controller „Car heating system"

Step 2b: Fuzzification of variables and values:

- For each variable and each value a Fuzzy Set is defined.
- Here is the Fuzzification of temperature. It is assumed that the normal temperature is around 20° C (imagine it as the temperature that is adjusted at controller of the heating).

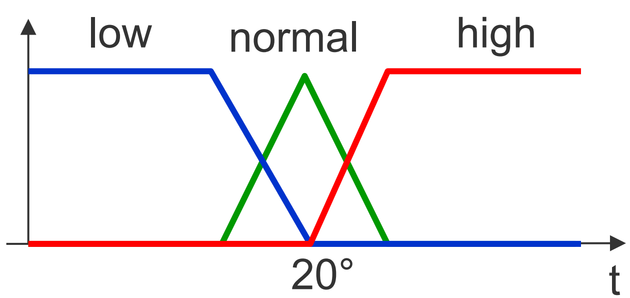
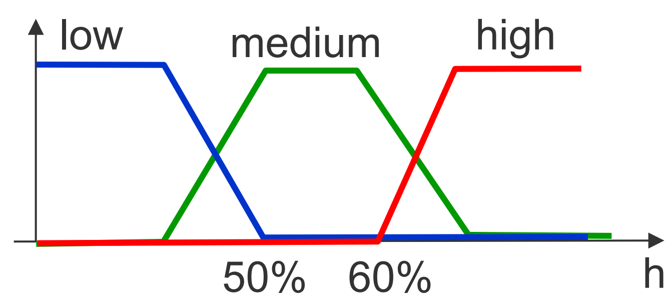
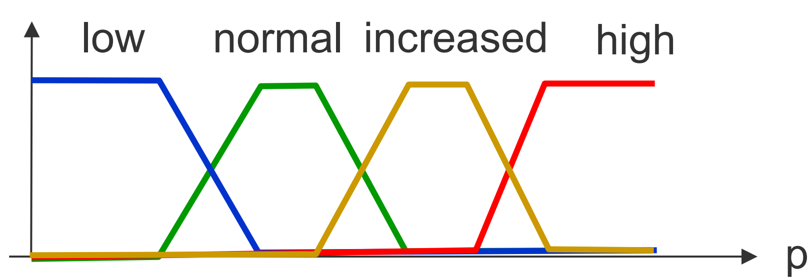


Fuzzy sets for temperature in one graphic



Fuzzy Logic Controller „Car heating system”

Step 2: Fuzzification of variables and values:

Input variables:	***Temperature***	***Humidity***
Fuzzy values:		
Output variable:	***Heating Power***	
Fuzzy values:		

Fuzzy Logic Controller „Car heating system"

Step 3: Define fuzzy IF-THEN rules

- A fuzzy IF-THEN rule is NOT a logical implication, but can be thought of as a command.
- A set of Fuzzy IF-THEN rules maps linguistic variables to lingustic variabes (fuzzy function).
- Fuzzy IF-THEN rules describe the control of the system. They are similar to the experiences of an expert, who would formulate their knowledge in natural language terms.

Fuzzy Logic Controller „Car heating system”

Step 3: Define fuzzy IF-THEN rules

- Rule 1:

  IF Temperature = *low*
  THEN heating power is *increased*

- Rule 2:

  IF Temperature = *normal* AND humidity = *low*
  THEN heating power is *normal*

- Rule 3:

  IF Temperature = *normal* AND humidity = *high*
  THEN heating power is *high*

- Rule 4:

  IF Temperature = *high*
  THEN heating power is *low*

Rule Application: „Car heating system”

Rule Application is performed in four steps:

1. Evaluate Antecedents:
   - For an exact input value, determine to which degree each antecedent is satisfied
   - Combine the degrees using the logical operators (AND in our example)
2. Evaluate Consequents:
   - The degree to wich an input variables A_i is satisfied determines the degree to which the corresponding output variable B_i holds (because IF-THEN rules are fuzzy functions). The result is the alpha cut of the output variable.
3. Aggregate Consequents:
   - Each rule gives one fuzzy set as a fuzzy output. Since all rules are valid, the fuzzy outputs may overlap (law of the excluded middle does not hold in general!). Combine them by OR to obtain a single fuzzy output value («aggregated output»).
4. Defuzzify Aggregated Output

Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

Assume the sensors have measured the following exact input values:

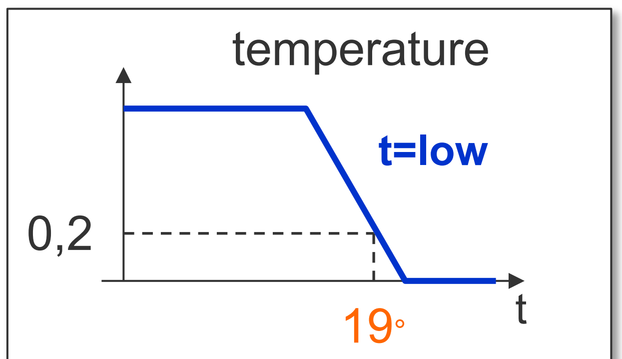
Temperature: t=19°
Humidity: h=45%

Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

Rule 1:

IF temperature = *low*
THEN heating power is *increased*

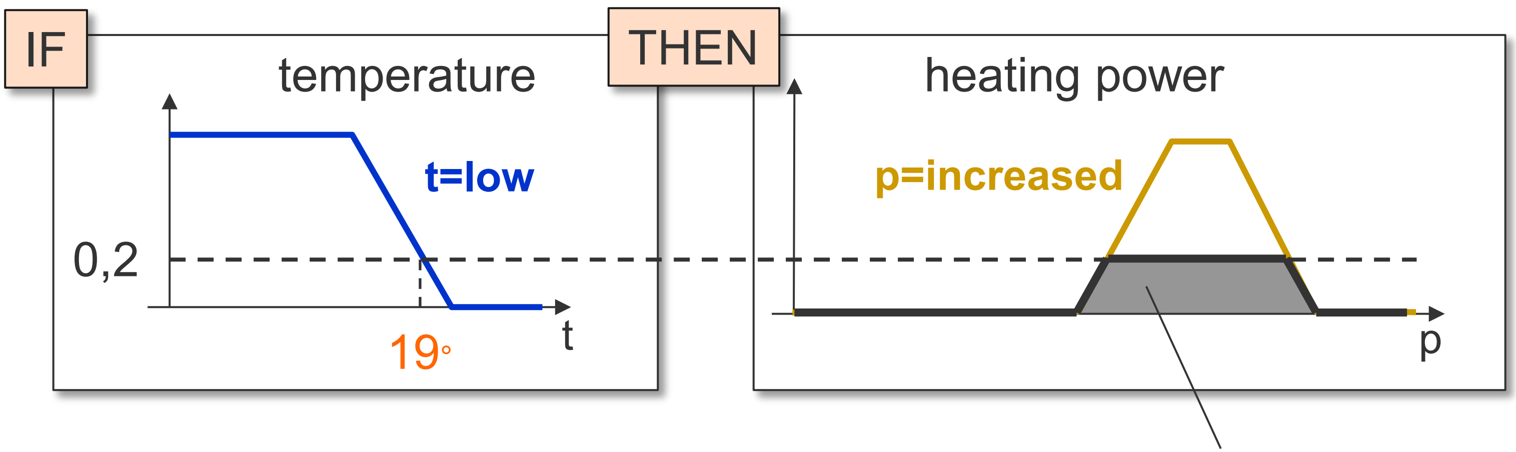


$$\mu_{t=low}(19°) = 0.2$$

Rule Application: „Car heating system"

Step 2: Evaluate Consequents

Rule 1: IF temperature = *low*
THEN heating power is *increased*



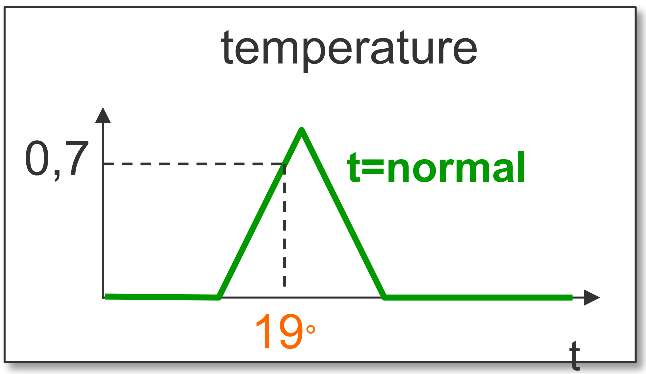
Output fuzzy set: $(\mu_{p=increased})_{0.2}$

(α-cut of $\mu_{p=increased}$ with $\alpha = 0.2$.)

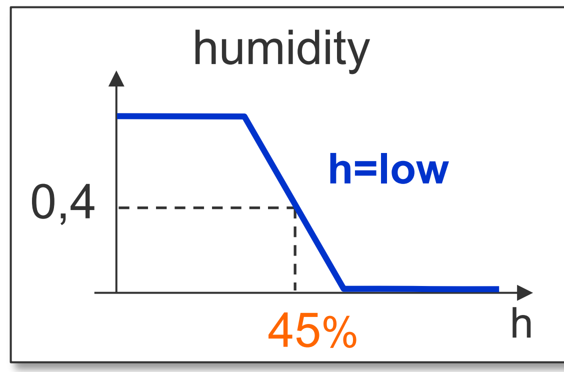
Rule Application: „Car heating system"

Step 1: Evaluate Antecedents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*



AND



$$\mu_{t=normal}(19°) = 0.7$$

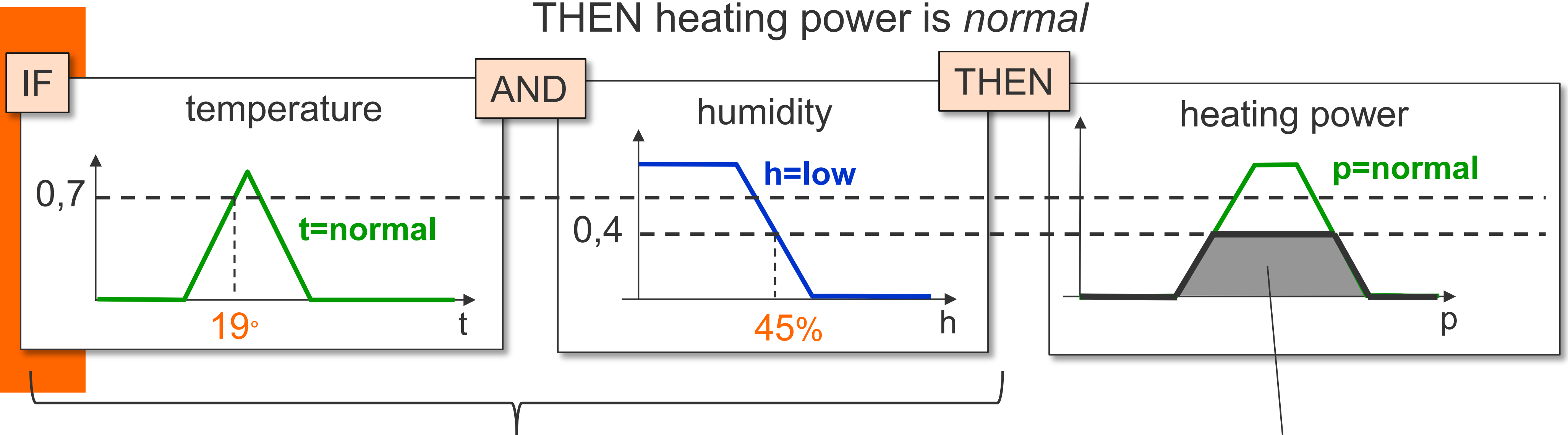
$$\mu_{h=low}(45\%) = 0.4$$

Min-Operator for AND: $\mu_{t=normal \wedge h=low}(19°, 45\%) = \min\{0.7,\ 0.4\} = 0.4$

Rule Application: „Car heating system”

Step 2: Evaluate Consequents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*



Min-Operator for AND:

$$\mu_{t=normal \wedge h=low}(19°,45\%)=0.4$$

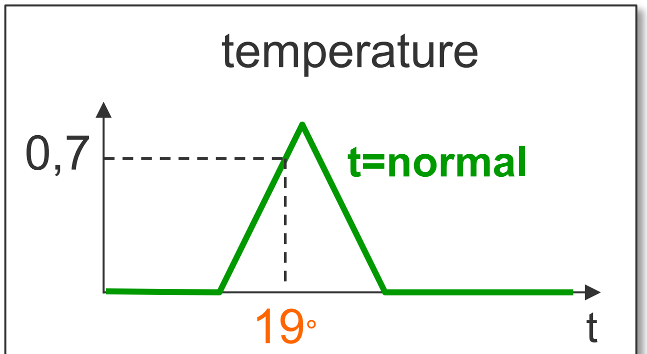
Output fuzzy set:

$$(\mu_{p=normal})_{0.4}$$

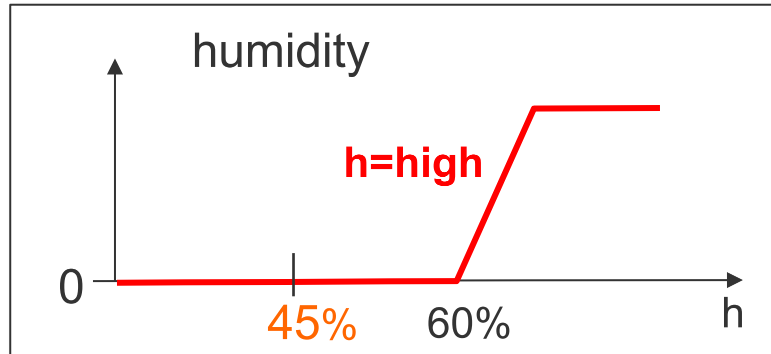
Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*



AND



$$\mu_{t=normal}(19°) = 0.7$$

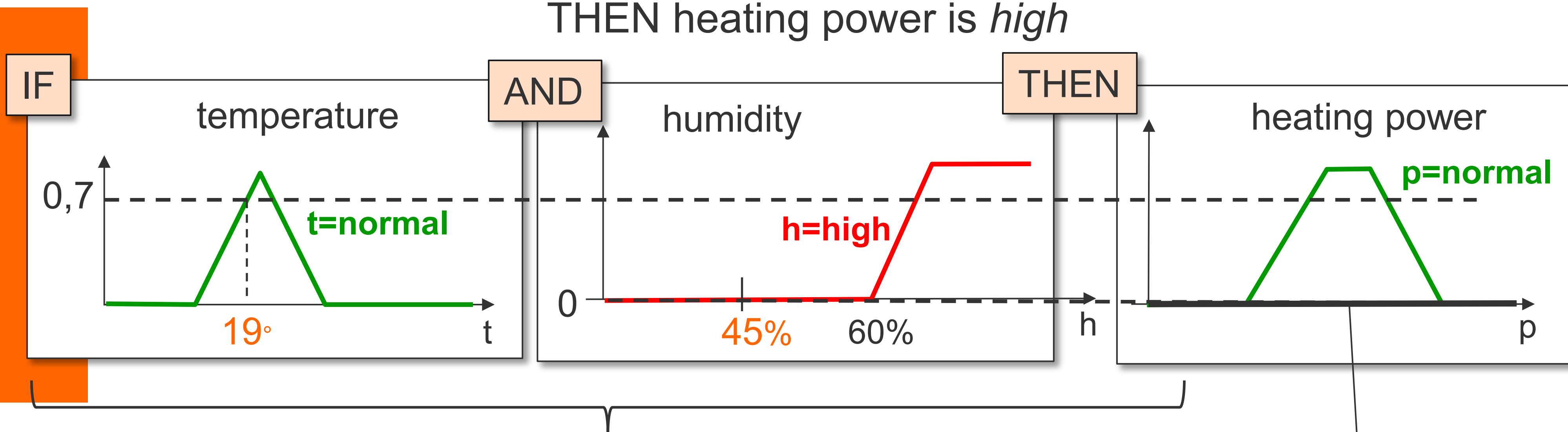
$$\mu_{h=high}(45\%) = 0$$

Min-Operator for AND: $$\mu_{t=normal \wedge h=high}(19°, 45\%) = \min\{0.7,\ 0\} = 0$$

Rule Application: „Car heating system"

Step 2: Evaluate Consequents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*



Min-Operator for AND:

$$\mu_{t=normal \wedge h=high}(19°,45\%)=0$$

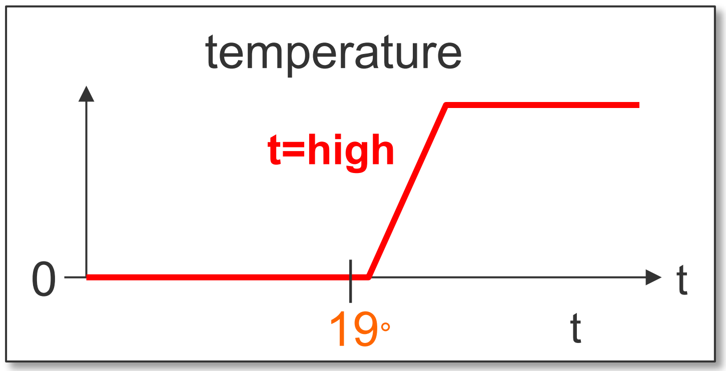
Output fuzzy set:

$$(\mu_{p=normal})_0 \equiv 0$$

Rule Application: „Car heating system"

Step 1: Evaluate Antecedents

Rule 4: IF Temperature = *high*
THEN heating power is *low*

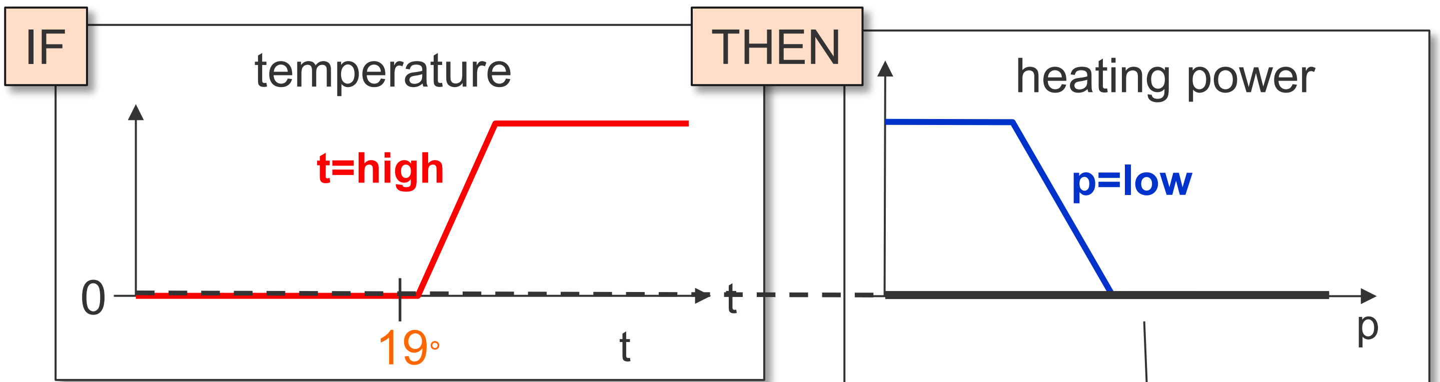


$$\mu_{t=high}(19°)=0$$

Rule Application: „Car heating system"

Step 2: Evaluate Consequents

Rule 4: IF Temperature = *high*
THEN heating power is *low*

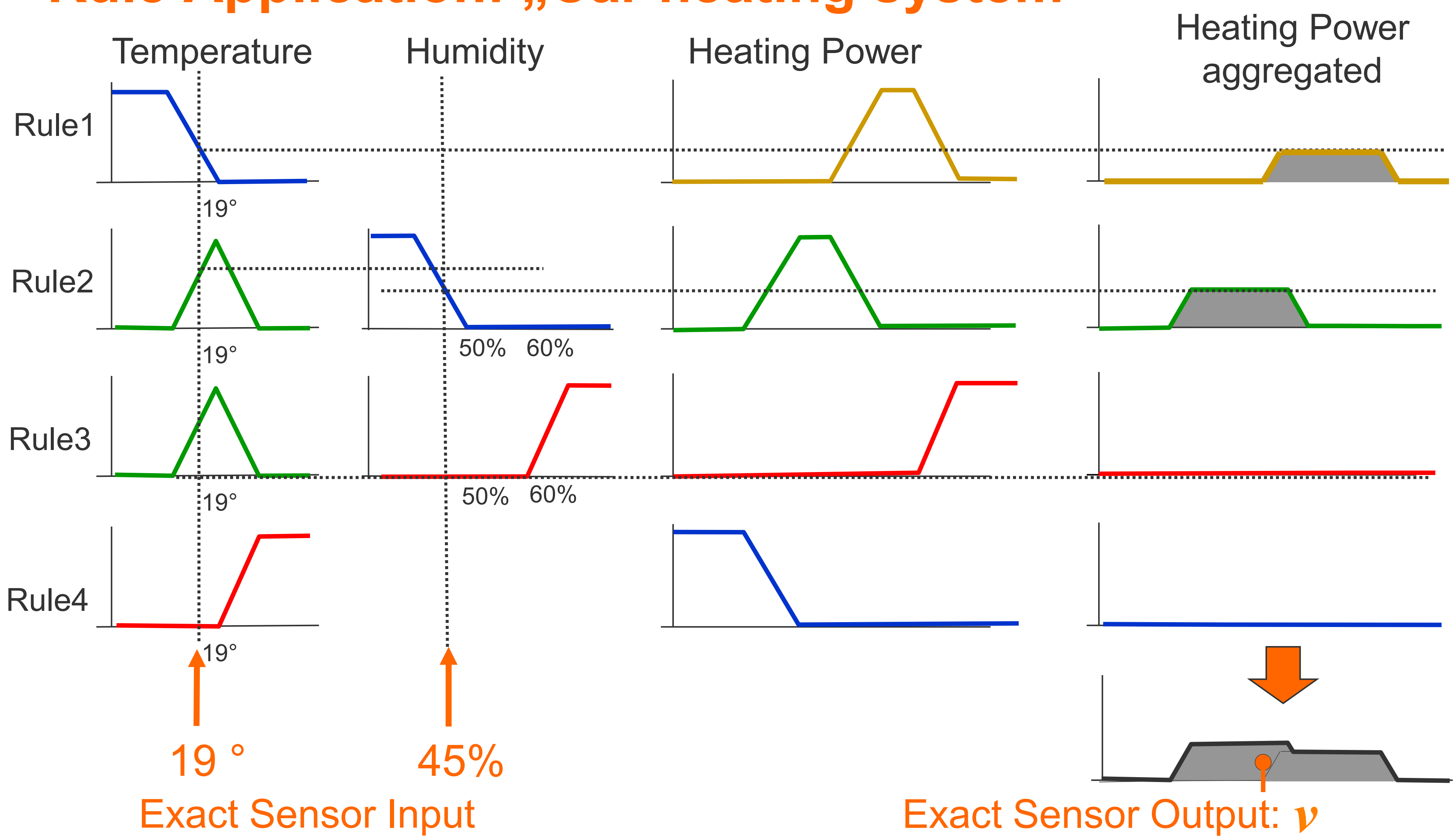


$$\mu_{t=high}(19°)=0$$

Output fuzzy set:

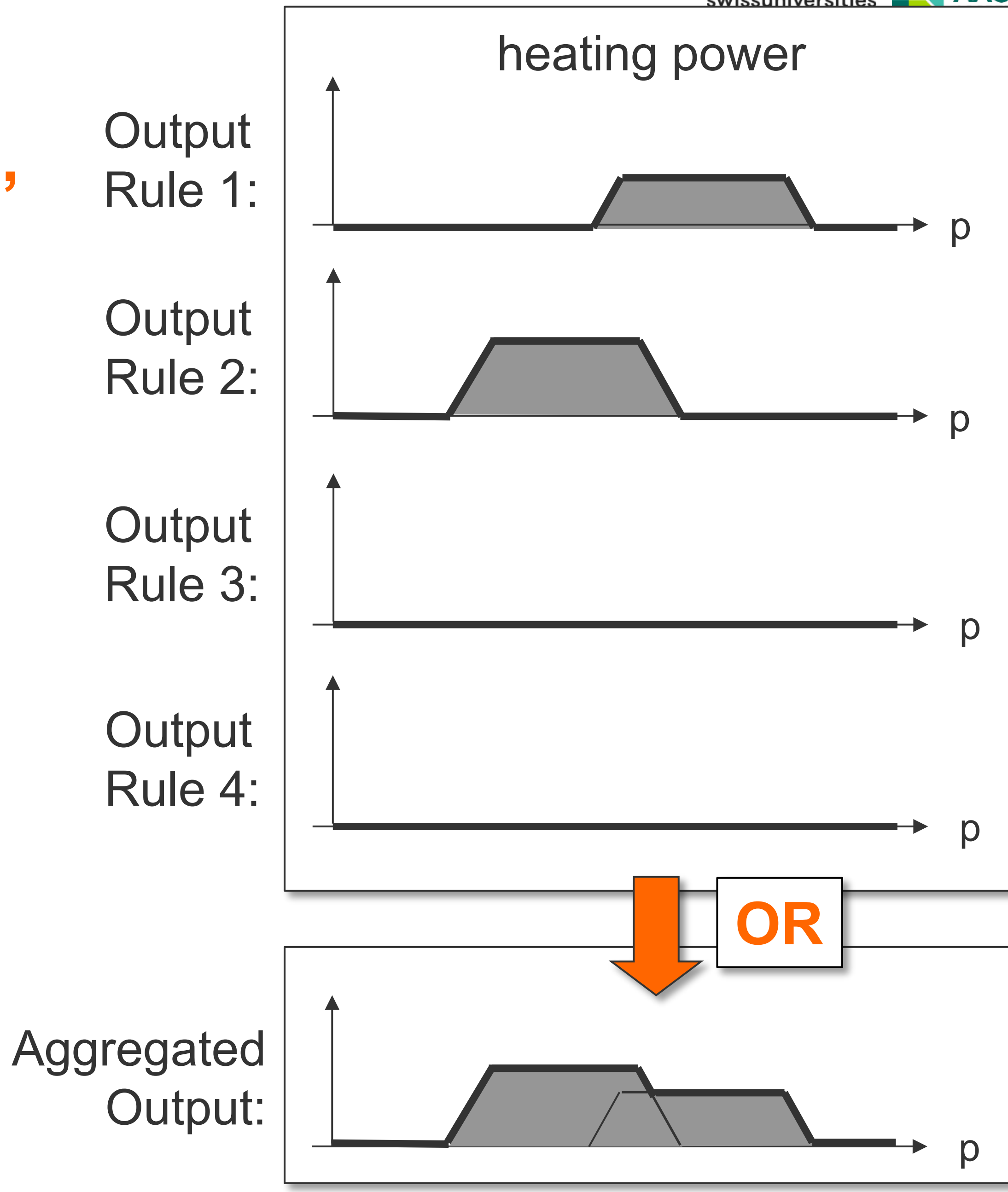
$$(\mu_{p=normal})_0 \equiv 0$$

Rule Application: „Car heating system"



Rule Application: „Car heating system"

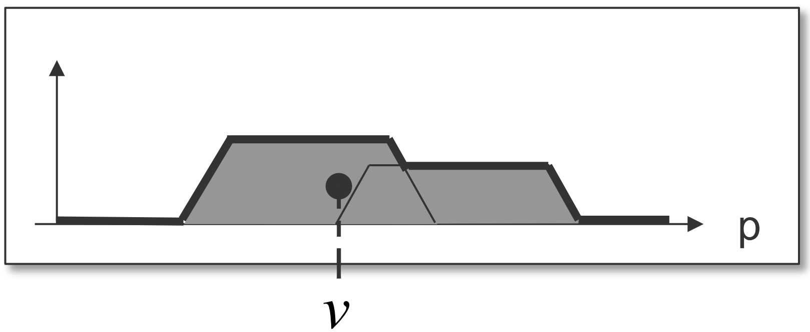
Step 3: Aggregate Evaluated Consequents:



Rule Application: „Car heating system”

Step 4: Defuzzify aggregated output

Center of gravity method:



$$v = \frac{\int \mu_A(x) \cdot x \, dx}{\int \mu_A(x) \, dx}$$

Rule Application: „Car heating system”

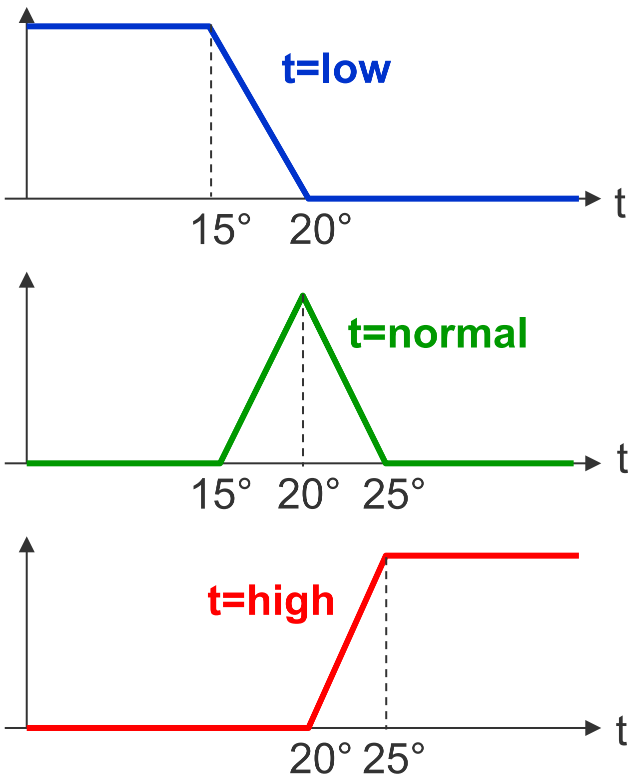
Main difference to exact reasoning:

Several rules can be active at the same time!
(Usually with different strengths.)

FUZZY LOGIC IN FLEX

Flex ist die Fuzzy-Komponent of LPA Prolog https://lpa.co.uk/

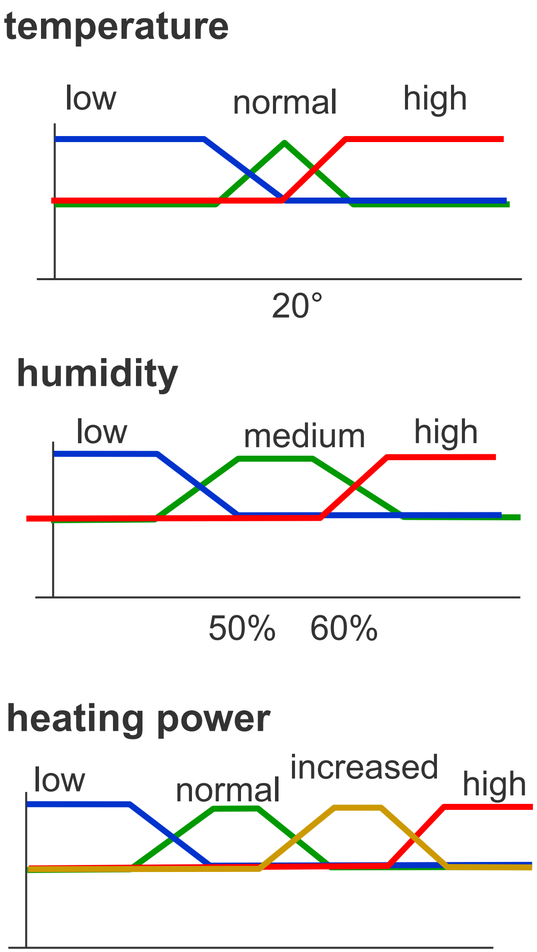
Representation of Fuzzy Logic in Flex




```
fuzzy_variable temperature ;
   ranges from -50 to 50 ;
   fuzzy_set low is \ shaped and linear at 15, 20 ;
   fuzzy_set normal is /\ shaped and linear at 15, 20, 25 ;
   fuzzy_set high is / shaped and linear at 20, 25 ;
```

Flex ist die Fuzzy-Komponent of LPA Prolog https://lpa.co.uk/

Repräsentation von Fuzzy-Logik in Flex




```
fuzzy_variable temperature ;
  ranges from -50 to 50 ;
  fuzzy_set low is \ shaped and linear at 15, 20 ;
  fuzzy_set normal is /\ shaped and linear at 15, 20, 15;
  fuzzy_set high is / shaped and linear at 20, 25 ;


fuzzy_variable humidity ;
  ranges from 0 to 100 ;
  fuzzy_set low is \ shaped and linear at 40, 50 ;
  fuzzy_set medium is /-\ shaped and linear at 40, 50, 60, 70;
  fuzzy_set high is / shaped and linear at 60, 100 ;


fuzzy_variable heatingpower ;
  ranges from 0 to 100 ;
  fuzzy_set low is \ shaped and linear at 20, 30 ;
  fuzzy_set normal is /-\ shaped and linear at 20, 30, 40, 50;
  fuzzy_set increased is /-\ shaped and linear at 40,50,60,70;
  fuzzy_set high is / shaped and linear at 60, 70 ;
```

Fuzzy-Regeln in Flex

IF Temperature = *low*
THEN heating power is *increased*

IF Temperature = *normal* AND humidity = *low*
THEN heating power is *normal*

IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*

IF Temperature = *high*
THEN heating power is *low*

```
fuzzy_rule r_1 if temperature is low
    then heatingpower is increased.

fuzzy_rule r_2 if temperature is normal and humidity is low
    then heatingpower is normal.

fuzzy_rule r_3 if temperature is normal and humidity is high
    then heatingpower is normal.

fuzzy_rule r_4 if temperature is high
    then heatingpower is low.
```